Quasi-Lie schemes: theory and applications

2009 J. Phys. A: Math. Theor. 42 335206

(http://iopscience.iop.org/1751-8121/42/33/335206)

J. Phys. A: Math. Theor. **42** (2009) 335206 (20pp) doi:10.1088/1751-8113/42/33/335206

Quasi-Lie schemes: theory and applications

José F Cariñena[1], Janusz Grabowski[2] and Javier de Lucas[1,2]

[1] Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain
[2] Institute of Mathematics, Polish Academy of Sciences, ul Śniadeckich 8, PO Box 21, 00-956 Warszawa, Poland

E-mail: jagrab@impan.pl

Received 3 April 2009, in final form 14 June 2009
Published 30 July 2009
Online at stacks.iop.org/JPhysA/42/335206

Abstract
A powerful method to solve nonlinear first-order ordinary differential equations, which is based on a geometrical understanding of the corresponding dynamics of the so-called Lie systems, is developed. This method enables us not only to solve some of these equations, but also gives geometrical explanations for some, already known, ad hoc methods of dealing with such problems.

PACS numbers: 02.30.Hq, 02.30.Jr, 02.40.−k
Mathematics Subject Classification: 34A26, 34A05, 34A34, 17B66, 22E70

1. Introduction

Systems of nonautonomous first-order ordinary differential equations appear often in Mathematics, Physics, Chemistry and Engineering, and therefore methods to solve them and analyse their properties are especially interesting because they allow us to understand many important problems in these various fields.

A special kind of these systems, the so-called *Lie systems* (or *Lie–Scheffers systems*) [1–5], has recently been analysed in many papers [6–14]. An important property of such systems is that they admit a certain set of time-dependent changes of variables which transforms each Lie system into a new one [10]. Such changes can be used to transform a given Lie system into an easily integrable one, i.e. into a Lie system related to a solvable Lie algebra of vector fields. In all these cases, we can obtain constants of the motion, integrability conditions or even solutions [14].

This transformation property is not only valid for Lie systems, but it still holds for a more general set of systems of differential equations. In order to generalize this property, we develop the concept of a *quasi-Lie system*, i.e. a system of nonautonomous differential equations accompanied with a flow of diffeomorphisms transforming it into a Lie system. We must stress that quasi-Lie systems are not equivalent to Lie systems in the trivial sense: the transformations we use are time dependent and we might get time-dependent superposition rules for them; meanwhile, Lie systems are related to superposition rules which are time independent. Furthermore, we define *quasi-Lie schemes*. A scheme is a procedure that, in some cases, enables us to transform a given nonautonomous system of differential equations into a Lie system. This scheme, when applied to Lie systems, gets back to the well-known transformations of Lie systems, see [7]. However, many other important systems of first-order differential equations can be studied by means of this new method. In this sense, the method of quasi-Lie schemes may be viewed as an abstract unification of many ad hoc developed integration methods existing in the literature.

The aim of this paper is to introduce and to analyse the properties of quasi-Lie schemes and to illustrate the theory by developing some interesting examples. More specifically, in order to the paper be self-contained, section 2 is devoted to review the theory of time-dependent vector fields. Then, in section 3, we summarize some properties of Lie systems and in section 4 we study the quasi-Lie scheme concept which may be used to relate certain sort of systems of differential equations to Lie systems. Such quasi-Lie schemes are applied in section 5 to analyse some interesting systems of differential equations. Some well-known differential equations are analysed: dissipative Milne–Pinney equations, nonlinear oscillator and Emden equations in order to recover some of their properties from a unified point of view. The field of applications of these differential equations is very broad, i.e. just the Emden equation appears in Mathematical Physics [15], Theoretical Physics [16], Astronomy [17], Astrophysics [18] and the review by Wong [19] contains about 140 references. We also study a time-dependent dissipative Mathews–Lakshmanan oscillator, and we provide a, as far as we know, new time-dependent constant of the motion. Finally, in section 6, we sum up the conclusions of our paper and give an outlook of possible problems to be studied by means of the methods developed here.

2. Generalized flows and time-dependent vector fields

A nonautonomous system of first-order ordinary differential equations in a manifold N is represented by a time-dependent vector field $X = X(t, x)$ on such a manifold. On a non-compact manifold, the vector field $X_t(x) = X(t, x)$, for a fixed t, is generally not defined globally, but it is well defined on a neighbourhood of each point $x_0 \in N$ for sufficiently small t. It is convenient to add time t to the manifold and to consider the *autonomization* of our system, i.e. the vector field

$$\bar{X}(t, x) = \frac{\partial}{\partial t} + X(t, x),$$

defined on a neighbourhood N^X of $\{0\} \times N$ in $\mathbb{R} \times N$. The vector field X_t is then defined on the open set of N,

$$N_t^X = \{x_0 \in N \mid (t, x_0) \in N^X\},$$

for all $t \in \mathbb{R}$. If $N_t^X = N$ for all $t \in \mathbb{R}$, we speak about a *global time-dependent vector field*. The system of differential equations associated with the time-dependent vector field $X(t, x)$ is written in local coordinates

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = X^i(t, x), \qquad i = 1, \ldots, n = \dim N,$$

where $X(t, x) = \sum_{i=1}^n X^i(t, x)\partial/\partial x^i$ is defined locally on the manifold for sufficiently small t.

A solution of this system is represented by a curve $s \mapsto \gamma(s)$ in N (integral curve) whose tangent vector $\dot{\gamma}$ at t, so at the point $\gamma(t)$ of the manifold, equals $X(t, \gamma(t))$. In other words,

$$\dot{\gamma}(t) = X(t, \gamma(t)). \tag{1}$$

It is well known that, at least for smooth X which we work with, for each x_0 there is a unique maximal solution $\gamma_X^{x_0}(t)$ of system (1) with the initial value x_0, i.e. satisfying $\gamma_X^{x_0}(0) = x_0$. This solution is defined at least for ts from a neighbourhood of 0. In case $\gamma_X^{x_0}(t)$ is defined for all $t \in \mathbb{R}$, we speak about a *global time solution*. The collection of all maximal solutions of system (1) gives rise to a (local) generalized flow g^X on N. By a *generalized flow* g on N we understand a smooth time-dependent family g_t of local diffeomorphisms on N, $g_t(x) = g(t, x)$, such that $g_0 = \mathrm{id}_N$. More precisely, g is a smooth map from a neighbourhood N^g of $\{0\}\times N$ in $\mathbb{R}\times N$ into N, such that g_t maps diffeomorphically the open submanifold $N_t^g = \{x_0 \in N \mid (t, x_0) \in N^g\}$ onto its image, and $g_0 = \mathrm{id}_N$. Again, for each $x_0 \in N$ there is a neighbourhood U_{x_0} of x_0 in N and $\epsilon > 0$ such that g_t is defined on U_{x_0} for $t \in (-\epsilon, \epsilon)$ and maps U_{x_0} diffeomorphically onto $g_t(U_{x_0})$.

If $N_t^g = N$ for all $t \in \mathbb{R}$, we speak about a *global generalized flow*. In this case, $g : t \in \mathbb{R} \mapsto g_t \in \mathrm{Diff}(N)$ may be viewed as a smooth curve in the diffeomorphism group $\mathrm{Diff}(N)$ with $g_0 = \mathrm{id}_N$.

Here it is also convenient to *autonomize* the generalized flow g extending it to a single local diffeomorphism

$$\bar{g}(t, x) = (t, g(t, x)) \tag{2}$$

defined on the neighbourhood N^g of $\{0\} \times N$ in $\mathbb{R} \times N$. The generalized flow g^X induced by the time-dependent vector field X is defined by

$$g^X(t, x_0) = \gamma_X^{x_0}(t). \tag{3}$$

Note that, for $g = g^X$, equation (3) can be rewritten in the form

$$X_t = X(t, x) = \dot{g}_t \circ g_t^{-1}. \tag{4}$$

In the above formula, we understood X_t and $\dot{g}_t$ as maps from N into TN, where $\dot{g}_t(x)$ is the vector tangent to the curve $s \mapsto g(s, x)$ at $g(t, x)$. Of course, the composition $\dot{g}_t \circ g_t^{-1}$, sometimes called the *right-logarithmic derivative* of $t \mapsto g_t$, is defined only for those points $x_0 \in N$ for which it makes sense. But it is always the case locally for sufficiently small t.

Let us observe that equation (4) defines, in fact, a one-to-one correspondence between generalized flows and time-dependent vector fields modulo the observation that the domains of $\dot{g}_t \circ g_t^{-1}$ and X_t need not coincide. In any case, however, $\dot{g}_t \circ g_t^{-1}$ and X_t coincide in a neighbourhood of any point for sufficiently small t. One can simply say that the *germs* of X and $\dot{g}_t \circ g_t^{-1}$ coincide, where the germ in our context is understood as the class of corresponding objects that coincide on a neighbourhood of $\{0\} \times N$ in $\mathbb{R} \times N$.

Indeed, for a given g, the corresponding time-dependent vector field is defined by (4). Conversely, for a given X, equation (4) determines the germ of the generalized flow $g(t, x)$ uniquely, as for each $x = x_0$ and for small t equation (4) implies that $t \mapsto g(t, x_0)$ is the solution of the system defined by X with the initial value x_0. In this way we get the following.

Theorem 1. *Equation (4) defines a one-to-one correspondence between the germs of generalized flows and the germs of time-dependent vector fields on N. For compact N, this correspondence reduces to a one-to-one correspondence between global time-dependent vector fields and global generalized flows.*

Any two generalized flows g and h can be composed: by definition $(g \circ h)_t = g_t \circ h_t$, where, as usual, we view $g_t \circ h_t$ as a local diffeomorphism defined for points for which the composition makes sense. It is important that in a neighbourhood of any point it really makes sense for sufficiently small t. As generalized flows correspond to time-dependent vector fields, this gives rise to an action of a generalized flow h on a time-dependent vector field X, giving rise to $h_\star X$ defined by the equation

$$g^{h_\star X} = h \circ g^X. \tag{5}$$

To obtain a more explicit form of this action, let us observe that

$$(h_\star X)_t = \frac{\mathrm{d}(h \circ g^X)_t}{\mathrm{d}t} \circ (h \circ g^X)_t^{-1} = \left(\dot{h}_t \circ g_t^X + Dh_t\left(\dot{g}_t^X\right)\right) \circ (g^X)_t^{-1} \circ h_t^{-1},$$

and therefore

$$(h_\star X)_t = \dot{h}_t \circ h_t^{-1} + Dh_t\left(\dot{g}_t^X \circ (g^X)_t^{-1}\right) \circ h_t^{-1},$$

i.e.

$$(h_\star X)_t = \dot{h}_t \circ h_t^{-1} + (h_t)_*(X_t), \tag{6}$$

where $(h_t)_*$ is the standard action of diffeomorphisms on vector fields. In a slightly different form, this can be written as an action of time-dependent vector fields on time-dependent vector fields:

$$(g^Y_\star X)_t = Y_t + \left(g_t^Y\right)_*(X_t). \tag{7}$$

For global time-dependent vector fields on compact manifolds, the latter defines a group structure in global time-dependent vector fields. This is an infinite-dimensional analogue of a group structure on paths in a finite-dimensional Lie algebra which has been used as a source for a nice construction of the corresponding Lie group in [20]. Since every generalized flow has the inverse, $(g^{-1})_t = (g_t)^{-1}$, so generalized flows, or better to say, the corresponding germs, form a group, and formula (7) allow us to compute the time-dependent vector field (right-logarithmic derivative) X_t^{-1} associated with the inverse. It is the time-dependent vector field

$$X_t^{-1} = -\left(g_t^X\right)_*^{-1}(X_t). \tag{8}$$

For time-independent vector fields $X_t = X_0$ for all t we have $(g_t^X)_* X = X$ and we also get the well-known formula

$$X^{-1} = -X.$$

Note that, by definition, the integral curves of $h_\star X$ are of the form $h_t(\gamma(t))$, where $\gamma(t)$ are integral curves of X. We can summarize our observation as follows.

Theorem 2. *Equation (6) defines a natural action of generalized flows on time-dependent vector fields. This action is a group action in the sense that*

$$(g \circ h)_\star X = g_\star(h_\star X).$$

The integral curves of $h_\star X$ are of the form $h_t(\gamma(t))$, for $\gamma(t)$ being an arbitrary integral curve for X.

The above action of generalized flows on time-dependent vector fields can also be defined in an elegant way by means of the corresponding autonomizations. It is namely easy to check the following.

Theorem 3. *For any generalized flow h and any time-dependent vector field X on a manifold N, the standard action $\bar{h}_* \bar{X}$ of the diffeomorphism $\bar{h}$, being the autonomization of h, on the vector field $\bar{X}$, being the autonomization of X, is the autonomization of the time-dependent vector field $h_\star X$:*

$$\bar{h}_* \bar{X} = \overline{h_\star X}.$$

3. Lie systems and superposition rules

The conditions for the system determined by a time-dependent vector field $X(t, x)$ on a manifold N ensuring that it admits a *superposition rule*, i.e. that there exists, at least locally for an open $U \subset N^m \times \mathbb{R}^n$, a map $\Phi : U \to N, x = \Phi(x_{(1)}, \ldots, x_{(m)}; k_1, \ldots, k_n)$, such that its general solution can be written as

$$x(t) = \Phi(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_n),$$

where $\{x_{(a)}(t) \mid a = 1, \ldots, m\}$ is any family of particular solutions 'in a general position' and $k = (k_1, \ldots, k_n)$ is a set of n arbitrary constants, which were studied by Lie [1]. Let us stress that the superposition function Φ is time independent.

The necessary and sufficient conditions say that the associated time-dependent vector field X can be written as a linear combination

$$X_t = \sum_{\alpha=1}^{r} b_\alpha(t) X_{(\alpha)}, \tag{9}$$

such that the vector fields $\{X_{(\alpha)} \mid \alpha = 1, \ldots, r\}$ generate a finite-dimensional real Lie algebra, the so-called Vessiot–Guldberg Lie algebra. The latter means that there exist r^3 real numbers $c_{\alpha\beta\gamma}$, such that

$$[X_{(\alpha)}, X_{(\beta)}] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} X_{(\gamma)}, \qquad \alpha, \beta = 1, \ldots, r.$$

Linear systems are particular instances of Lie systems associated with a Vessiot–Guldberg Lie algebra, isomorphic to the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ and $m = n$ in the homogeneous case, or the corresponding affine algebra and $m = n + 1$ in the inhomogeneous one. The Riccati equation is another example for which $X_t = b_0(t)Y_0 + b_1(t)Y_1 + b_2(t)Y_2$ with

$$Y_0 = \frac{\partial}{\partial x}, \qquad Y_1 = x\frac{\partial}{\partial x}, \qquad Y_2 = x^2\frac{\partial}{\partial x},$$

closing on a Vessiot–Guldberg Lie algebra isomorphic to the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, see [7, 9]. There is an action $\Phi_{\rm Ric} : SL(2, \mathbb{R}) \times \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ of the Lie group $SL(2, R)$ on $\bar{\mathbb{R}} \equiv \mathbb{R} \cup \infty$ [7] such that the fundamental vector fields of this action are linear combinations with real coefficients of the vector fields Y_0, Y_1 and Y_2.

Another relevant example of a Lie system is given by a time-dependent right-invariant vector field in a Lie group G. If $\{\mathrm{a}_1, \ldots, \mathrm{a}_r\}$ is a basis of T_eG and $X^{\rm R}_\alpha$ are the corresponding right-invariant vector fields, $X^{\rm R}_\alpha(g) = (R_g)_*\mathrm{a}_\alpha$, then the time-dependent right-invariant vector field

$$X_t = -\sum_{\alpha=1}^{r} b_\alpha(t) X^{\rm R}_\alpha,$$

defines a Lie system in G whose integral curves are solutions of the system $\dot{g} = -\sum_{\alpha=1}^{r} b_\alpha(t) X^{\rm R}_\alpha(g)$. Applying $(R_{g^{-1}(t)})_*$ to both sides, we see that $g(t)$ satisfies

$$(R_{g^{-1}(t)})_*\dot{g}(t) = -\sum_{\alpha=1}^{r} b_\alpha(t)\mathrm{a}_\alpha \in T_eG. \tag{10}$$

Right invariance means that it is enough to know one solution, for instance the one starting from the neutral element e, to know all the solutions of the equation with any initial condition, i.e. we obtain the solution $g'(t)$ with the initial condition $g'(0) = g_0$ as $R_{g(0)}g(t)$. A generalization of the method used by Wei and Norman for linear systems [21] is very useful in solving such equations and furthermore there exist reduction techniques that can also be used [11]. Finally, as right-invariant vector fields X^{R} project onto the fundamental vector fields in each homogeneous space of G, the solution of (10) allows us to find the general solution for the corresponding Lie system in each homogeneous space. Conversely, the knowledge of particular solutions of the associated system in a homogeneous space gives us a method for reducing the problem to the corresponding isotropy group [11]. This equation is also important because any Lie system described by a time-dependent vector field on a manifold N, like (9), where the vector fields are complete and satisfy the same commutation relations as the basis $\{\mathrm{a}_1, \ldots, \mathrm{a}_r\}$, determines an action $\Phi_{\mathrm{LieG}} : G \times N \to N$ such that the vector field X_α is the fundamental vector field corresponding to a_α, and moreover, the integral curves for the time-dependent vector field are obtained from the solutions of equation (10). More explicitly, the general solution of the given Lie system is $x(t) = \Phi_{\mathrm{LieG}}(g(t), x_0)$, where x_0 is the initial condition of the solution and $g(t)$ is the solution of equation (10) with $g(0) = e$.

The search for the number m of solutions and the superposition function Φ has recently been studied from a geometric perspective [12]. Essentially, we should consider 'diagonal prolongations' to N^{m+1}, $\widetilde{X}(x_{(0)}, \ldots, x_{(m)}, t)$, of the time-dependent vector field $X(t, x) = \sum_{i=1}^n X^i(t, x)\partial/\partial x^i$, given by

$$\widetilde{X}(x_{(0)}, \ldots, x_{(m)}, t) = \sum_{a=0}^{m} X_a(x_{(a)}, t), \qquad t \in \mathbb{R},$$

where $X_a(x_{(a)}, t) = \sum_{i=1}^n X^i(x_{(a)}, t)\partial/\partial x^i_{(a)}$, such that the extended system admits n independent constants of the motion, which define the superposition function in an implicit way.

4. Quasi-Lie systems and schemes

Definition 1. *By a quasi-Lie system, we understand a pair (X, g) consisting of a time-dependent vector field X on a manifold N (the system) and a generalized flow g on N (the control) such that $g_\star X$ is a Lie system.*

Since for the Lie system $g_\star X$ we are able to produce the general solution out of a number of known particular solutions, the knowledge of the control makes a similar procedure for our initial system possible. Indeed, let $\Phi = \Phi(x_1, \ldots, x_m, k_1, \ldots, k_n)$ be a superposition function for the Lie system $g_\star X$, so that, knowing m solutions $\bar{x}_{(1)}, \ldots, \bar{x}_{(m)}$ of $g_\star X$, we can derive the general solution of the form

$$\bar{x}_{(0)} = \Phi(\bar{x}_{(1)}, \ldots, \bar{x}_{(m)}, k_1, \ldots, k_n).$$

If we now know m independent solutions, $x_{(1)}, \ldots, x_{(m)}$ of X, then, according to theorem 3, $\bar{x}_a(t) = g_t(x_a(t))$ are solutions of $g_\star X$, producing a general solution of $g_\star X$ in the form $\Phi(\bar{x}_{(1)}, \ldots, \bar{x}_{(m)}, k_1, \ldots, k_n)$. It is now clear that

$$x_{(0)}(t) = g_t^{-1} \circ \Phi(g_t(x_{(1)}(t)), \ldots, g_t(x_{(m)}(t)), k_1, \ldots, k_n) \tag{11}$$

is a general solution of X. In this way, we have obtained a *time-dependent superposition rule* for the system X. We can summarize the above considerations as follows.

Theorem 4. *Any quasi-Lie system (X, g) admits a time-dependent superposition rule of form (11), where Φ is a superposition function for the Lie system $g_\star X$.*

Of course, the above time-dependent superposition rule is practically meaningful in finding the general solution of a system X only if the generalized flow g is explicitly known. An alternative abstract definition of a quasi-Lie system as a time-dependent vector field X for which there exists a generalized flow g such that $g_\bigstar X$ is a Lie system does not have much sense, as every X would be a quasi-Lie system in this context. For instance, given a time-dependent vector field X, the pair $(X, (g^X)^{-1})$ is a quasi–Lie system because $(g^X)_t^{-1} \circ g_t^X = \mathrm{id}_N$, thus $(g^X)_\bigstar^{-1} X = 0$, which is a Lie system trivially. On the other hand, finding $(g^X)^{-1}$ is nothing but solving our system X completely, so we just reduce to our original problem. In practice, it is therefore crucial that the control g comes from a system which can be integrated effectively. There are, however, many cases when our procedure works well and provides a geometrical interpretation of many, originally developed ad hoc, methods of integration. Consider, for instance, the following scheme that can lead to 'nice' quasi-Lie systems.

Take a finite-dimensional real vector space V of vector fields on N and consider the family of all time-dependent vector fields X on N such that X_t belongs to V on its domain, i.e. $X_t \in V_{|N_t^X}$. We will say that these are time-dependent vector fields taking values in V. The time-dependent vector fields taking values in V depend on a finite family of control functions. For example, take a basis $\{X_1, \ldots, X_r\}$ of V and consider a general time-dependent system with values in V determined by $b = b(t) = (b_1(t), \ldots, b_r(t))$ as

$$(X^b)_t = b_j(t) X_j.$$

On the other hand, the nonautonomous systems of differential equations associated with $X \in V|_{N_t^X}$ are not Lie systems in general, if V is not a Lie algebra itself. If we have additionally a finitely parametrized family of local diffeomorphism, say $\underline{g} = \underline{g}(a_1, \ldots, a_k)$, then any curve $a = a(t) = (a_1(t), \ldots, a_k(t))$ in the control parameters, defined for small t, gives rise to a generalized flow $g_t^a = \underline{g}(a(t))$. Let us assume additionally that there is a Lie algebra V_0 of vector fields contained in V. We can look for control functions $a(t)$ such that for certain $b(t)$ we get that $g_\bigstar^a X^b$ has values in V_0 for each time t. Let us denote this as

$$g_\bigstar^a X^b \in V_0. \tag{12}$$

This choice of control functions makes (X^b, g^a) into a quasi-Lie system, so we get time-dependent superposition rules for the corresponding systems X^b.

Let us observe that in the case when all the generalized flows g^a preserve V, i.e. for each time-dependent vector field $X^b \in V$ also $g_\bigstar^a X^b \in V$, the inclusion (12) becomes a differential equation for the control functions $a(t)$ in terms of the functions $b(t)$. This situation is not so rare as it may seem to be at the first sight. Suppose, for instance, that we find a Lie algebra $W \subset V$ such that $[W, V] \subset V$ and that the time-dependent vector fields with values in W can be effectively integrated to generalized flows. In this case, any time-dependent vector field Y^a with values in W gives rise to a generalized flow g^a which, in view of transformation rule (7), preserves the set of time-dependent vector fields with values in V. For each $b = b(t)$, the inclusion (12) becomes therefore a differential equation for the control function $a = a(t)$ which often can be effectively solved.

Definition 2. *Let W and V be finite-dimensional real vector spaces of vector fields on a manifold N. We say that they form a quasi-Lie scheme $S(W, V)$ if the following are satisfied:*

(1) W is a vector subspace of V.
(2) W is a Lie algebra of vector fields, i.e. $[W, W] \subset W$.
(3) W normalizes V, i.e. $[W, V] \subset V$.

If V is a Lie algebra of vector fields V, we call the quasi-Lie scheme $S(V, V)$ simply a Lie scheme $S(V)$.

Remark 1. There is the largest Lie subalgebra we can use as W—the normalizer of V in V. Sometimes, however, it is useful to consider smaller Lie subalgebras W.

We say that a time-dependent vector field X is in a quasi-Lie scheme $S(W, V)$, and write $X \in S(W, V)$, if X belongs to V on its domain, i.e. $X_t \in V_{|N_t^X}$. Note that, by definition, the set of all time-dependent vector fields belonging to $S(W, V)$ depends only on V, and the choice of W is irrelevant.

Now, given a quasi-Lie scheme $S(W, V)$ which we will call sometimes simply a scheme, we may consider the group $\mathcal{G}(W)$ of generalized flows associated with W.

Definition 3. *We call the group of the scheme $S(W, V)$ the group $\mathcal{G}(W)$ of generalized flows corresponding to the time-dependent vector fields with values in W.*

Proposition 1. (Main property of a scheme) *Given a scheme $S(W, V)$, a time-dependent vector field $X \in S(W, V)$ and a generalized flow $g \in \mathcal{G}(W)$, we get that $g_\bigstar X \in S(W, V)$.*

The proof is obvious and follows from the fact that if Y is a generalized flow with values in W and X takes values in V, then, according to formula (7), $g^Y_\bigstar X$ takes values in V as well, as $[W, V] \subset V$ and V is finite dimensional.

From the last definition, we can state the definition of a quasi-Lie system with respect to a scheme.

Definition 4. *Given a quasi-Lie scheme $S(W, V)$ and a time-dependent vector field $X \in S(W, V)$, we say that X is a quasi-Lie system with respect to $S(W, V)$ if there exists a generalized flow $g \in \mathcal{G}(W)$ and a Lie algebra of vector fields $V_0 \subset V$ such that*

$$g_\bigstar X \in S(V_0).$$

We emphasize that if X is a quasi-Lie system with respect to the scheme $S(W, V)$, it automatically admits a time-dependent superposition rule in the form given by (11). In the following section, we apply our theory and illustrate these concepts with examples.

5. Applications of quasi-Lie schemes

The above-mentioned properties of quasi-Lie schemes and quasi-Lie systems can be used to investigate some previously studied systems of differential equations [22–33] systematically from this new perspective.

In this section, we apply quasi-Lie schemes to study dissipative Milne–Pinney equations, nonlinear oscillators and Emden differential equations. More precisely, we first apply our theory to dissipative Milne–Pinney equations. These systems cannot be treated with the theory of Lie systems directly, but one can use a quasi-Lie scheme to transform them into Milne–Pinney equations. These latter equations have been proved to be SODE Lie systems recently [6, 13], and this fact enables us to get time-dependent superposition rules for dissipative Milne–Pinney equations by means of the superposition rule found for non-dissipative ones.

Next, we analyse nonlinear oscillators. Perelomov studied some of these systems in order to relate them to other important systems [28]. The cases investigated by Perelomov were selected and obtained by means of ad hoc methods. Here we approach some instances treated in [28] in order to explain Perelomov's work from the point of view of the theory of quasi-Lie schemes. As a result, we show how our theory provides time-dependent constants of the motion and clarify some points about this work.

We also analyse Emden equations. In this case, we use a quasi-Lie scheme to obtain constants of the motion for Emden equations whose time-dependent coefficients satisfy certain conditions. Notwithstanding, this is just a small instance of what can be made by means of our scheme.

Finally, we analyse certain kind of dissipative Mathews–Lakshmanan oscillators [31–33]. Some kinds of these nonlinear oscillators have been recently investigated from the point of view of classical mechanics [33]. Here we just perform a simple application of the theory of quasi-Lie schemes to relate different types of dissipative Mathews–Lakschmanan oscillators to the usual ones.

Let us provide a quasi-Lie scheme to deal with some of the systems investigated in following subsections. Recall that we need to find vector spaces W and V of vector fields satisfying the three conditions stated in definition 2. Consider the vector space V spanned by the linear combinations of the vector fields

$$X_1 = x\frac{\partial}{\partial v}, \quad X_2 = x^n\frac{\partial}{\partial v}, \quad X_3 = v\frac{\partial}{\partial x}, \quad X_4 = v\frac{\partial}{\partial v}, \quad X_5 = x\frac{\partial}{\partial x} \tag{13}$$

on $\mathrm{T}\mathbb{R}$ and take the vector subspace $W \subset V$ generated by

$$Y_1 = X_4 = v\frac{\partial}{\partial v}, \quad Y_2 = X_1 = x\frac{\partial}{\partial v}, \quad Y_3 = X_5 = x\frac{\partial}{\partial x}.$$

Therefore, W is a solvable Lie algebra of vector fields,

$$[Y_1, Y_2] = -Y_2, \quad [Y_1, Y_3] = 0, \quad [Y_2, Y_3] = -Y_2,$$

and taking into account that

$$\begin{aligned} &[Y_1, X_2] = -X_2, \qquad &[Y_1, X_3] = X_3, \qquad &[Y_2, X_2] = 0,\\ &[Y_2, X_3] = X_5 - X_4, \qquad &[Y_3, X_2] = nX_2, \qquad &[Y_3, X_3] = -X_3, \end{aligned}$$

we see that V is invariant under the action of W, i.e. $[W, V] \subset V$. In this way, we get the quasi-Lie scheme $S(W, V)$. We stress that the vector space V is not a Lie algebra because the commutator $[X_2, X_3]$ is not in V. Moreover, there is no Lie algebra of vector fields $V' \supseteq V$ and thus V cannot be related to a Lie scheme.

The key tool provided by the scheme $S(W, V)$ is the infinite-dimensional group $\mathcal{G}(W)$ of generalized flows for the time-dependent vector fields with values in W, i.e. $\alpha_1(t)Y_1 + \alpha_2(t)Y_2 + \alpha_3(t)Y_3$. The integration of such t-dependent vector fields leads to the description of the time-dependent changes of variables associated with $\mathcal{G}(W)$, i.e.

$$\mathcal{G}(W) = \left\{ g(\alpha(t), \beta(t), \gamma(t)) \right.$$
$$\left. = \begin{cases} x = \gamma(t)x' \\ v = \alpha(t)v' + \beta(t)x' \end{cases} \middle| \alpha(t), \gamma(t) > 0, \alpha(0) = \gamma(0) = 1, \beta(0) = 0 \right\}.$$

5.1. Dissipative Milne–Pinney equations

In this section, we study the so-called dissipative Milne–Pinney equations. We show that the first-order ordinary differential equations associated with these second-order ones in the usual way, i.e. by considering velocities as new variables, are not Lie systems. However, the theory of quasi-Lie schemes can be used to deal with such first-order systems. Here we provide a scheme which enables us to transform a certain kind of dissipative Milne–Pinney equations, considered as first-order systems, into some first-order Milne–Pinney equations already studied by means of the theory of Lie systems [6]. As a result, we get a time-dependent superposition rule for some of these dissipative Milne–Pinney equations.

Let us state the problem under study. Consider the family of dissipative Milne–Pinney equations of the form

$$\ddot{x} = a(t)\dot{x} + b(t)x + c(t)\frac{1}{x^3}. \tag{14}$$

We are mainly interested in the case $c(t) \neq 0$, so we assume that $c(t)$ has a constant sign for the set of values of t we are considering.

Usually, we associate with such a second-order differential equation a system of first-order differential equations by introducing a new variable v and relating the differential equation (14) to the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = a(t)v + b(t)x + c(t)\dfrac{1}{x^3}. \end{cases} \tag{15}$$

In order to verify that the quasi-Lie scheme $S(W, V)$, for the case $n = -3$, can be used to handle the latter system, that is $X \in S(W, V)$, we have to ensure that the time-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + \left(a(t)v + b(t)x + \frac{c(t)}{x^3}\right)\frac{\partial}{\partial v},$$

whose integral curves are solutions for system (15), is such that $X_t \in V$ for every t in an open interval in $\mathbb{R}$. In this way, in view of (13), we observe that

$$X_t = a(t)X_4 + b(t)X_1 + c(t)X_2 + X_3,$$

and thus $X \in S(W, V)$. Moreover $V'' = \langle X_1, \ldots, X_4\rangle$ is not a Lie algebra of vector fields because $[X_3, X_2] \notin V''$. Also, there is no finite-dimensional real Lie algebra V' containing V''. Thus, system (15) is not a Lie system but we can use the quasi-Lie scheme $S(W, V)$ to investigate it.

Let us consider the infinite-dimensional subgroup of $\mathcal{G}(W)$ given by its time-dependent changes of variables with $\gamma(t) = 1$. According to the general theory of quasi-Lie schemes, these time-dependent changes of variables enable us to transform system (15) into a new one again describing the integral curves for a time-dependent vector field $X' \in S(W, V)$, that is

$$X'_t = a'(t)X_4 + b'(t)X_1 + c'(t)X_2 + d'(t)X_3 + e'(t)X_5. \tag{16}$$

The new coefficients are

$$\begin{cases} a'(t) = a(t) - \beta(t) - \dfrac{\dot{\alpha}(t)}{\alpha(t)}, \\ b'(t) = \dfrac{b(t)}{\alpha(t)} + a(t)\dfrac{\beta(t)}{\alpha(t)} - \dfrac{\beta^2(t)}{\alpha(t)} - \dfrac{\dot{\beta}(t)}{\alpha(t)}, \\ c'(t) = \dfrac{c(t)}{\alpha(t)}, \\ d'(t) = \alpha(t), \\ e'(t) = \beta(t). \end{cases}$$

The integral curves for the time-dependent vector field (16) are solutions of the system

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t} = \beta(t)x' + \alpha(t)v', \\ \dfrac{\mathrm{d}v'}{\mathrm{d}t} = \left(\dfrac{b(t)}{\alpha(t)} + a(t)\dfrac{\beta(t)}{\alpha(t)} - \dfrac{\beta^2(t)}{\alpha(t)} - \dfrac{\dot{\beta}(t)}{\alpha(t)}\right)x' \\ \qquad + \left(a(t) - \beta(t) - \dfrac{\dot{\alpha}(t)}{\alpha(t)}\right)v' + \dfrac{c(t)}{\alpha(t)}\dfrac{1}{x'^3}. \end{cases} \tag{17}$$

As it was said in Section 4, we use schemes to transform the corresponding systems of first-order differential equations into Lie ones. So, in this case, we must find a Lie algebra of vector fields $V_0 \subset V$ and a generalized flow $g \in \mathcal{G}(W)$ such that $g_\bigstar X \in S(V_0)$. This leads to a system of ordinary differential equations for the functions $\alpha(t)$, $\beta(t)$ and some integrability conditions about the initial functions $a(t)$, $b(t)$ and $c(t)$ for such a time-dependent change of variables to exist.

In order to find a proper Lie algebra of vector fields $V_0 \subset V$, note that Milne–Pinney equations studied in [6] are Lie systems in the family of differential equations defined by systems (15) and therefore it is natural to look for the conditions needed to transform a given system (15), described by the time-dependent vector field X_t, into one of these first-order Milne–Pinney equations of the form

$$\begin{cases} \dot{x} = f(t)v, \\ \dot{v} = -\omega(t)x + f(t)\dfrac{k}{x^3}, \end{cases} \tag{18}$$

where k is a constant, i.e. a system describing the integral curves for a time-dependent vector field with values in the Lie algebra of vector fields [6]

$$V_0 = \langle X_3 + kX_2, X_1, \tfrac{1}{2}(X_5 - X_4)\rangle.$$

As a result, we get that $\beta = 0$, $\alpha = f$ and, furthermore, the functions α, a and c must satisfy

$$k\alpha^2 = c, \qquad \dot{\alpha} - a\alpha = 0, \tag{19}$$

so that c must be of a constant sign equal to that of k. The second condition is a differential equation for α and the first one determines c in terms of α. Therefore, both conditions lead to a relation between c and a providing the integrability condition

$$c(t) = k\exp\left(2A(t)\right), \quad \text{and} \quad A(t) = \int a(t)\mathrm{d}t, \tag{20}$$

and showing, in view of (17)–(19), that

$$\alpha(t) = \exp\left(A(t)\right) \quad \text{and} \quad \omega(t) = -b(t)\exp\left(-A(t)\right),$$

where we choose the constants of integration in order to get $\alpha(0) = 1$ as required.

Summing up the preceding results, under the integrability condition (20), the first-order Milne–Pinney equation (15) can be transformed into the system

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t} = \exp\left(A(t)\right)v', \\ \dfrac{\mathrm{d}v'}{\mathrm{d}t} = b(t)\exp\left(-A(t)\right)x' + \exp\left(A(t)\right)\dfrac{k}{x'^3}, \end{cases}$$

by means of the time-dependent change of variables

$$g\left(\exp\left(A(t)\right), 0, 1\right) = \begin{cases} x' = x, \\ v' = \exp\left(A(t)\right)v. \end{cases}$$

Note 5.1. The previous change of variables is a particular instance of the so-called Liouville transformation [34].

Now, the final Milne–Pinney equation can be rewritten by means of the time reparametrization

$$\tau(t) = \int \exp\left(A(t)\right)\mathrm{d}t,$$

as

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}\tau} = v', \\ \dfrac{\mathrm{d}v'}{\mathrm{d}\tau} = \exp\left(-2A(t)\right) b(t(\tau))x' + \dfrac{k}{x'^3}. \end{cases}$$

These systems were analysed in [7], and there it was shown through the theory of Lie systems that they admit the constant of the motion

$$I = (\bar{x}v' - \bar{v}x')^2 + k\left(\frac{\bar{x}}{x'}\right)^2,$$

where $(\bar{x}, \bar{v})$ is a solution of the system

$$\begin{cases} \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}\tau} = \bar{v}, \\ \dfrac{\mathrm{d}\bar{v}}{\mathrm{d}\tau} = \exp\left(-2A(t)\right) b(t)\bar{x}, \end{cases}$$

which can be written as a second-order differential equation

$$\frac{\mathrm{d}^2\bar{x}}{\mathrm{d}\tau^2} = \exp\left(-2A(t)\right) b(t)\bar{x}.$$

If we invert the time reparametrization, we obtain the following differential equation:

$$\ddot{\bar{x}} - a(t)\dot{\bar{x}} - b(t)\bar{x} = 0, \qquad (21)$$

which is the linear differential equation associated with the initial Milne–Pinney equation.

As it was shown in [6], we can obtain, by means of the theory of Lie systems, the following superposition rule:

$$x' = \frac{\sqrt{2}}{|\bar{x}_1\bar{v}_2 - \bar{v}_1\bar{x}_2|}\left(I_2\bar{x}_1^2 + I_1\bar{x}_2^2 \pm \sqrt{4I_1I_2 - k(\bar{x}_1\bar{v}_2 - \bar{v}_1\bar{v}_2)^2}\bar{x}_1\bar{x}_2\right)^{1/2},$$

and as the time-dependent transformation performed does not change the variable x, we get the time-dependent superposition rule

$$x = \frac{\sqrt{2}\alpha(t)}{|\bar{x}_1\dot{\bar{x}}_2 - \dot{\bar{x}}_1\bar{x}_2|}\left(I_2\bar{x}_1^2 + I_1\bar{x}_2^2 \pm \sqrt{4I_1I_2 - \frac{k}{\alpha^2(t)}(\bar{x}_1\dot{\bar{x}}_2 - \dot{\bar{x}}_1\bar{x}_2)^2}\bar{x}_1\bar{x}_2\right)^{1/2}$$

in terms of a set of solutions of the second-order linear system (21).

Summing up, the application of our scheme to the family of dissipative Milne–Pinney equations

$$\ddot{x} = a(t)\dot{x} + b(t)x + \exp\left(2\int a(t)\mathrm{d}t\right)\frac{k}{x^3}$$

shows that it admits a time-dependent superposition principle,

$$x = \frac{\sqrt{2}\alpha(t)}{|y_1\dot{y}_2 - y_2\dot{y}_1|}\left(I_2y_1^2 + I_1y_2^2 \pm \sqrt{4I_1I_2 - \frac{k}{\alpha^2(t)}(y_1\dot{y}_2 - y_2\dot{y}_1)^2}y_1y_2\right)^{1/2},$$

in terms of two independent solutions y_1 and y_2 for the differential equation

$$\ddot{y} - a(t)\dot{y} - b(t)y = 0.$$

So, we have fully detailed a particular application of the theory of quasi-Lie schemes to dissipative Milne–Pinney equations. As a result, we provide a time-dependent superposition rule for a family of such systems. Another paper dealing with such an approach to dissipative Milne–Pinney equations and explaining some of their properties can be found in [35].

5.2. Nonlinear oscillators

As a second application of our theory, we use quasi-Lie schemes to deal with a certain kind of nonlinear oscillators. The main objective of this section is to explain some properties of some time-dependent nonlinear oscillators studied by Perelomov in [28]. We also furnish with, as far as we know, a new constant of the motion for these systems.

Consider the subset of the family of nonlinear oscillators investigated in [28]:

$$\ddot{x} = b(t)x + c(t)x^n, \qquad n \neq 0, 1.$$

The cases $n = 0, 1$ are omitted because they can be handled with the usual theory of Lie systems. Like in the above section, we link the above second-order ordinary differential equation to the first-order system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = b(t)x + c(t)x^n. \end{cases} \tag{22}$$

Now, we have to go over whether the solutions of system (22) are integral curves for a time-dependent vector field $X \in S(W, V)$. In order to check out this, we realize that system (22) describes the integral curves for the time-dependent vector field

$$X_t = v\frac{\partial}{\partial x} + (b(t)x + c(t)x^n)\frac{\partial}{\partial v},$$

which can be written as

$$X_t = b(t)X_1 + c(t)X_2 + X_3. \tag{23}$$

Note also that $[X_2, X_3] \notin V$ and not only $V'' = \langle X_1, X_2, X_3\rangle$ is not a Lie algebra of vector fields but also there is no finite-dimensional Lie algebra V' including V''. Thus, X cannot be considered as a Lie system and we conclude that the first-order nonlinear oscillator (22) describing integral curves of the time-dependent vector field (23) (which is not a Lie system) can be described by means of the quasi-Lie scheme $S(W, V)$.

Let us restrict ourselves to analyse those time-dependent changes of variables associated with the generalized flows of $\mathcal{G}(W)$ with $\beta(t) = \dot{\gamma}(t)$ and $\alpha(t) = 1/\gamma(t)$ and apply these transformations to system (22). The main theorem of the theory of quasi-Lie systems tells us that

$$g(\alpha(t), \beta(t), \gamma(t))_\bigstar X \in S(W, V).$$

Indeed, these time-dependent transformations lead to the systems

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t} = \dfrac{1}{\gamma^2(t)}v', \\ \dfrac{\mathrm{d}v'}{\mathrm{d}t} = (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n, \end{cases} \tag{24}$$

which are related to the second-order differential equations

$$\gamma^2(t)\ddot{x}' = -2\gamma(t)\dot{\gamma}(t)\dot{x}' + (\gamma^2(t)b(t) - \ddot{\gamma}(t)\gamma(t))x' + c(t)\gamma^{n+1}(t)x'^n.$$

But the theory of quasi-Lie schemes is based on finding a generalized flow $g \in \mathcal{G}(W)$ such that $g_\bigstar X$ becomes a Lie system, i.e. there exists a Lie algebra of vector fields $V_0 \subset V$ such that $g_\bigstar X \in S(V_0)$. For instance, we can try to transform a particular instance of systems (24) into a first-order differential equation associated with a nonlinear oscillator with a zero time-dependent angular frequency, for example, into the first-order system

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}t} = f(t)v', \\ \dfrac{\mathrm{d}v'}{\mathrm{d}t} = f(t)c_0x'^n, \end{cases} \tag{25}$$

related to the nonlinear oscillator

$$\frac{\mathrm{d}^2x'}{\mathrm{d}\tau^2} = c_0 x'^n,$$

with $\mathrm{d}\tau/\mathrm{d}t = f(t)$.

The conditions ensuring such a transformation are

$$\gamma(t)b(t) - \ddot{\gamma}(t) = 0, \qquad c(t) = c_0\gamma^{-(n+3)}(t), \tag{26}$$

with $f(t) = \gamma_1^{-2}(t)$, where γ_1 is a non-vanishing particular solution for $\gamma(t)b(t) - \ddot{\gamma}(t) = 0$. We must emphasize that just particular solutions with $\gamma_1(0) = 1$ and $\dot{\gamma}_1(0) = 0$ are related to generalized flows in $\mathcal{G}(W)$. Nevertheless, any other particular solution can also be used to transform a nonlinear oscillator into a Lie system as we stated. The Lie system (25) is the system associated with the time-dependent vector field

$$X_t = \frac{1}{\gamma_1^2(t)}\left(v'\frac{\partial}{\partial x'} + c_0 x'^n\frac{\partial}{\partial v'}\right).$$

As a consequence of the standard methods developed for the theory of Lie systems [13], we join two copies of the above system in order to get the first integrals

$$I_i = \frac{1}{2}v_i'^2 - \frac{c_0}{n+1}x_i'^{n+1}, \qquad i = 1, 2,$$

and

$$I_3 = \frac{x_1'}{\sqrt{I_1}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_1'^{n+1}}{I_1(n+1)}\right) - \frac{x_2'}{\sqrt{I_2}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_2'^{n+1}}{I_2(n+1)}\right),$$

where $\mathrm{Hyp}(a, b, c, d)$ denotes the corresponding hypergeometric functions. In terms of the initial variables, these first integrals for $g_\star X$ read

$$I_i = \frac{1}{2}(\gamma_1(t)\dot{x}_i - \dot{\gamma}_1(t)x_i)^2 - \frac{c_0}{\gamma_1^{n+1}(t)(n+1)}x_i^{n+1}, \qquad i = 1, 2, \tag{27}$$

and

$$I_3 = \frac{1}{\gamma_1(t)}\left(\frac{x_1}{\sqrt{I_1}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_1^{n+1}}{\gamma_1^{n+1}(t)I_1(n+1)}\right) - \frac{x_2}{\sqrt{I_2}}\mathrm{Hyp}\left(\frac{1}{n+1}, \frac{1}{2}, 1 + \frac{1}{n+1}, -\frac{c_0 x_2^{n+1}}{\gamma_1^{n+1}(t)I_2(n+1)}\right)\right). \tag{28}$$

As a particular application of conditions (26), we can consider the following example of [28], where the time-dependent Hamiltonian:

$$H(t) = \frac{1}{2}p^2 + \frac{\omega^2(t)}{2}x^2 + c^2\gamma_1^{-(s+2)}(t)x^s,$$

with γ_1 being such that $\ddot{\gamma}_1(t) + \omega^2(t)\gamma_1(t) = 0$ is studied. The Hamilton equations for the latter Hamiltonian are

$$\begin{cases} \dot{x} = p, \\ \dot{p} = -sc^2\gamma_1^{-(s+2)}(t)x^{s-1} - \omega^2(t)x, \end{cases} \tag{29}$$

being associated with the second-order differential equation for the variable x given by

$$\ddot{x} = -sc^2\gamma_1^{-(s+2)}(t)x^{s-1} - \omega^2(t)x. \tag{30}$$

The latter differential equations are particular cases of our Emden equations with

$$b(t) = -\omega^2(t), \qquad c(t) = -sc^2\gamma_1^{-(s+2)}(t), \quad n = s-1. \tag{31}$$

Note that here the variable p plays the role of v in our theoretical development. It can be easily verified that these coefficients satisfy conditions (26). Therefore, we get that the time-dependent frequency nonlinear oscillator (30) can be transformed into a new one with zero frequency, i.e.

$$\frac{\mathrm{d}^2x'}{\mathrm{d}\tau^2} = -sc^2x'^{s-1},$$

with

$$\tau = \int \frac{\mathrm{d}t}{\gamma_1^2(t)},$$

reproducing the result given by Perelomov [28]. The choice of the time-dependent frequencies is such that it is possible to transform the initial time-dependent nonlinear oscillator into the final autonomous nonlinear oscillator. Then, we recover here such frequencies as a result of an integrability condition. Moreover, in view of expressions (27), (28) and (31), we get, as far as we know, new time-dependent constants of the motion for these nonlinear oscillators.

5.3. The Emden equation

In this section, we apply quasi-Lie schemes to Emden equations. These equations appear broadly in the literature and have many applications; indeed, the review by Wong in 1977 [19] contains more than 100 cites. Here they are analysed in order to recover, under some integrability conditions, a set of time-dependent constants of the motion.

The Emden equations we investigate are

$$\ddot{x} = a(t)\dot{x} + b(t)x^n, \quad n \neq 1. \tag{32}$$

The case $n = 1$ is removed because it can be treated directly by means of the theory of Lie systems.

Emden equations are associated with the system of first-order differential equations

$$\begin{cases} \dot{x} = v, \\ \dot{v} = a(t)v + b(t)x^n. \end{cases} \tag{33}$$

As in the preceding examples, let us verify that the scheme $S(W, V)$ can be used to handle this system. System (33) describes the integral curves for the time-dependent vector field given by

$$X_t = v\frac{\partial}{\partial x} + (a(t)v + b(t)x^n)\frac{\partial}{\partial v},$$

which, in terms of the basis (13) for V, reads

$$X_t = a(t)X_4 + X_3 + b(t)X_2,$$

so that $X \in S(W, V)$. We must remark that, as $[X_3, X_2] \notin V$, there is no Lie algebra of vector fields containing the vector space V'' spanned by X_2, X_3 and X_4, and X_t cannot be considered as a Lie system.

The time-dependent change of variables induced by a control $g \in \mathcal{G}(W)$ transforms system (33) into

$$\left\{\begin{aligned}\frac{\mathrm{d}x'}{\mathrm{d}t} &= \left(\frac{\beta(t)}{\gamma(t)} - \frac{\dot\gamma(t)}{\gamma(t)}\right)x' + \frac{\alpha(t)}{\gamma(t)}v',\\ \frac{\mathrm{d}v'}{\mathrm{d}t} &= \left(a(t) - \frac{\beta(t)}{\gamma(t)} - \frac{\dot\alpha(t)}{\alpha(t)}\right)v' + \left(a(t)\frac{\beta(t)}{\alpha(t)} - \frac{\beta^2(t)}{\alpha(t)\gamma(t)} - \frac{\dot\beta(t)}{\alpha(t)} + \frac{\beta(t)\dot\gamma(t)}{\alpha(t)\gamma(t)}\right)x'\\ &\quad + \frac{b(t)\gamma^n(t)}{\alpha(t)}x'^n.\end{aligned}\right. \tag{34}$$

According to Theorem 1, the latter systems describe the integral curves of the time-dependent vector field $g_\star X \in S(W, V)$. Now, we must look for a Lie algebra of vector fields $V_0 \subset V$ and a control $g \in \mathcal{G}(W)$ such that $g_\star X \in S(V_0)$.

For the sake of simplicity, let us suppose that $\beta(t) = 0$. Thus, system (34) leads to

$$\left\{\begin{aligned}\frac{\mathrm{d}x'}{\mathrm{d}t} &= -\frac{\dot\gamma(t)}{\gamma(t)}x' + \frac{\alpha(t)}{\gamma(t)}v',\\ \frac{\mathrm{d}v'}{\mathrm{d}t} &= \left(a(t) - \frac{\dot\alpha(t)}{\alpha(t)}\right)v' + b(t)\frac{\gamma^n(t)}{\alpha(t)}x'^n.\end{aligned}\right.$$

We impose some conditions ensuring that this differential equation is a solvable Lie system. For instance, we want it to be of the form

$$\left\{\begin{aligned}\frac{\mathrm{d}x'}{\mathrm{d}t} &= f(t)\left(c_{11}x' + c_{12}v'\right),\\ \frac{\mathrm{d}v'}{\mathrm{d}t} &= f(t)\left(c_{21}x'^n + c_{22}v'\right),\end{aligned}\right. \tag{35}$$

where the coefficients c_{ij} are constant. Therefore, we get

$$f(t)c_{11} = -\frac{\dot\gamma(t)}{\gamma(t)}, \qquad f(t)c_{12} = \frac{\alpha(t)}{\gamma(t)},$$
$$f(t)c_{21} = \frac{b(t)\gamma^n(t)}{\alpha(t)}, \qquad f(t)c_{22} = a(t) - \frac{\dot\alpha(t)}{\alpha(t)},$$

that implies

$$\alpha(t) = -\frac{c_{12}}{c_{11}}\dot\gamma(t) \Longrightarrow -\frac{\ddot\gamma(t)}{\dot\gamma(t)} + a(t) = -\frac{c_{22}}{c_{11}}\frac{\dot\gamma(t)}{\gamma(t)}.$$

If we fix $c_{22} = 1$, $c_{11} = -1$, $c_{12} = 4$ and $c_{21} = -1$, and we define $A(t) = \int a(t)\mathrm{d}t$, the values of γ and α are

$$\gamma(t) = \sqrt{2\int \exp\left(A(t)\right)\mathrm{d}t}, \qquad \alpha(t) = \frac{4}{\gamma(t)}\exp\left(A(t)\right),$$

with appropriately chosen constants of integration to get $\gamma(0) = \alpha(0) = 1$ and $g(\alpha(t), 0, \gamma(t)) \in \mathcal{G}(W)$. Nevertheless, any other particular solutions with different initial conditions can also be used. Now, since

$$-\frac{b(t)\gamma^n(t)}{\alpha(t)} = \frac{1}{4}\frac{\alpha(t)}{\gamma(t)},$$

we see that

$$-b(t)\gamma^{n+3}(t) = 4\exp\left(2A(t)\right)$$

and

$$b^{-\frac{2}{n+3}}(t)\exp\left(\frac{4A(t)}{n+3}\right) - 2^{\frac{n-1}{n+1}}\int \exp(A(t))\,\mathrm{d}t = 0,$$

which is equivalent to the expression found in [27] if we do not fix the initial conditions for $\gamma(t)$ and $\alpha(t)$.

Now, let us obtain a constant of the motion for (35). As $\alpha(t)/\gamma(t) = 4f(t)$, we get

$$f(t) = \frac{\exp(A(t))}{2\int \exp(A(t))\,\mathrm{d}t}.$$

Hence, the system (35) admits a constant of the motion in the form

$$I = -2v'^2 - \frac{x'^{n+1}}{n+1} + x'v'.$$

If we invert the initial change of variables, we reach the following constant of the motion for our initial differential equation:

$$I' = \left(v^2 - \frac{2b(t)}{n+1}x^{n+1}\right)\exp(-2A(t))\int \exp(A(t))\,\mathrm{d}t - xv\exp(-A(t)),$$

which is equivalent to the one found by Sarlet and Bahar in [29].

5.4. Dissipative Mathews–Lakshmanan oscillators

In this section, we provide a simple application of the theory of quasi-Lie schemes to investigate the time-dependent dissipative Mathews–Lakshmanan oscillator

$$(1+\lambda x^2)\ddot{x} - F(t)(1+\lambda x^2)\dot{x} - (\lambda x)\dot{x}^2 + \omega(t)x = 0, \qquad \lambda > 0. \tag{36}$$

More specifically, we supply some integrability conditions to relate it to the Mathews–Lakshmanan oscillator [31, 32, 33, 36]

$$(1+\lambda x^2)\ddot{x} - (\lambda x)\dot{x}^2 + kx = 0, \qquad \lambda > 0, \tag{37}$$

and by means of such a relation we get, as far as we know, a new time-dependent constant of the motion.

Consider the system of a first-order differential equation related to equation (36) in the usual way, i.e.

$$\begin{cases} \dot{x} = v, \\ \dot{v} = F(t)v + \dfrac{\lambda x v^2}{1+\lambda x^2} - \omega(t)\dfrac{x}{1+\lambda x^2}, \end{cases} \tag{38}$$

and determining the integral curves for the time-dependent vector field

$$X_t = \left(F(t)v + \frac{\lambda x v^2}{1+\lambda x^2} - \omega(t)\frac{x}{1+\lambda x^2}\right)\frac{\partial}{\partial v} + v\frac{\partial}{\partial x}.$$

Let us provide a scheme to handle system (38). Consider the vector space V spanned by the vector fields

$$X_1 = v\frac{\partial}{\partial x} + \frac{\lambda x v^2}{1+\lambda x^2}\frac{\partial}{\partial v}, \qquad X_2 = \frac{x}{1+\lambda x^2}\frac{\partial}{\partial v}, \qquad X_3 = v\frac{\partial}{\partial v}, \tag{39}$$

and the linear space $W = \langle X_3\rangle$. The commutator relations

$$[X_3, X_1] = X_1, \qquad [X_3, X_2] = -X_2,$$

imply that the linear spaces W, V made up a quasi-Lie scheme $S(W, V)$. As the time-dependent vector field X_t reads in terms of the basis (39)

$$X_t = F(t)X_3 - \omega(t)X_2 + X_1,$$

we get that $X_t \in S(W, V)$.

The integration of X_3 shows that

$$\mathcal{G}(W) = \left\{ g(\alpha(t)) = \begin{cases} x = x', \\ v = \alpha(t)v'. \end{cases} \middle| \alpha(t) > 0, \alpha(0) = 1 \right\},$$

and the time-dependent changes of variables related to the controls of $\mathcal{G}(W)$ transform system (38) into

$$\begin{cases} \dot{x}' = \alpha(t)v', \\ \dot{v}' = \left(F(t) - \dfrac{\dot{\alpha}(t)}{\alpha(t)}\right)v' - \dfrac{\omega(t)}{\alpha(t)}\dfrac{x'}{1+\lambda x'^2} + \alpha(t)\dfrac{\lambda x'v'^2}{1+\lambda x'^2}. \end{cases}$$

Suppose that we fix $\dot{\alpha} - F(t)\alpha = 0$. Hence, the latter becomes

$$\begin{cases} \dot{x}' = \alpha(t)v', \\ \dot{v}' = -\dfrac{\omega(t)}{\alpha(t)}\dfrac{x'}{1+\lambda x'^2} + \alpha(t)\dfrac{\lambda x'v'^2}{1+\lambda x'^2}. \end{cases}$$

Let us try to search conditions for ensuring the above system to determine the integral curves for a time-dependent vector field of the form $X(t, x) = f(t)X(x)$ with $X \in V$, e.g.

$$\begin{cases} \dot{x}' = f(t)v', \\ \dot{v}' = f(t)\left(\dfrac{x'}{1+\lambda x'^2} + \dfrac{\lambda x'v'^2}{1+\lambda x'^2}\right). \end{cases}$$

In such a case, $\alpha(t) = f(t), \omega(t) = -\alpha^2(t)$ and therefore $\omega(t) = -\exp(2\int F(t)\mathrm{d}t)$. The time reparametrization $\mathrm{d}\tau = f(t)\mathrm{d}t$ transforms the previous system into the autonomous one

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}\tau} = v', \\ \dfrac{\mathrm{d}v'}{\mathrm{d}\tau} = \dfrac{x'}{1+\lambda x'^2} + \dfrac{\lambda x'v'^2}{1+\lambda x'^2} \end{cases}$$

determining the integral curves for the vector field $X = X_1 + X_2$ and related to a Mathews–Lakshmanan oscillator (37) with $k = 1$. The method of characteristics shows after a brief calculation that this system has a first integral

$$I(x', v') = \frac{1+\lambda x'^2}{1+\lambda v'^2},$$

that reads in terms of the initial variables and the time as a time-dependent constant of the motion

$$I(t, x, v) = \frac{\alpha^2(t) + \lambda\alpha^2(t)x^2}{\alpha^2(t) + \lambda v^2},$$

for the time-dependent dissipative Mathews–Lakshmanan oscillator (36) getting, as far as we know, a new t-dependent constant of the motion.

6. Conclusions and outlook

We develop the theory of quasi-Lie schemes as a generalization of the theory of Lie systems, and we prove some of their fundamental properties and find applications. In particular, we recover a time-dependent superposition rule for a family of dissipative Milne–Pinney equations. This result, which can be found in [37], is seen here from a new perspective as a result of a systematic treatment of the family of dissipative Milne–Pinney equations admitting such a time-dependent superposition rule.

Additionally, we explain from a geometric point of view some transformation properties of time-dependent nonlinear oscillators. More precisely, we provide a geometrical understanding for some results of the Perelomov's paper [28]. Moreover, the quasi-Lie approach allows us to investigate quantum nonlinear Hamiltonians and supply an explanation of the transformation properties for the quantum analogue of this physical model. Finally, we also recover time-dependent constants of the motion for the Emden equations and certain new dissipative time-dependent Mathews–Lakshmanan oscillator.

We hope that this shows that the theory of quasi-Lie schemes can be viewed as a good approach to study many interesting systems of differential equations and quantum Hamiltonians from the same geometric viewpoint. We follow this idea in some works under development [35].

Finally, the here developed examples prove that the set of time-dependent changes of variables allowing us to transform a differential equation in a scheme into a Lie system can be broader than those detailed for the group of such a scheme. We have already found an explanation for this fact which will be included in next works within the theory of quasi-Lie schemes.

Acknowledgments

Partial financial supports by research projects MTM2006-10531 and E24/1 (DGA) and by the Polish Ministry of Science and Higher Education under the grant no N201 005 31/0115 are acknowledged. JdL also acknowledges an FPU grant from Ministerio de Educación y Ciencia.

References

[1] Lie S and Scheffers G 1893 *Vorlesungen über continuierliche Gruppen mit geometrischen und anderen Anwendungen* ed G Scheffers (Leipzig: Teubner)

[2] Anderson R L, Harnad J and Winternitz P 1981 Group theoretical approach to superposition systems of Riccati equations *Lett. Math. Phys.* **5** 143–8

[3] Harnad J, Winternitz P and Anderson R L 1983 Superposition principles for matrix Riccati equations *J. Math. Phys.* **24** 1062–72

[4] Shnider S and Winternitz P 1984 Classification of systems of nonlinear ordinary differential equations with superposition principles *J. Math. Phys.* **25** 3155–65

[5] Winternitz P 1983 Lie groups and solutions of nonlinear differential equations *Nonlinear Phenomena (Lecture Notes in Physics* vol 189*)* ed K B Wolf (Berlin: Springer)

[6] Cariñena J F, de Lucas J and Rañada M F 2007 Nonlinear superpositions and Milne–Pinney equations. *Differential Geometric Methods in Mechanics and Field Theory* ed F Cantrijn, M Crampin and B Langerock (Gent: Academia Press) pp 15–33

[7] Cariñena J F, Ramos A and de Lucas J 2007 A geometric approach to integrability conditions for Riccati equations *Electron. J. Differ. Equ.* **122** 1

[8] Cariñena J F, Marmo G and Nasarre J 1998 The non-linear superposition principle and the Wei–Norman method *Int. J. Mod. Phys.* **13** 3601–27

[9] Cariñena J F and Ramos A 1999 Integrability of the Riccati equation from a group theoretical viewpoint *Int. J. Mod. Phys.* A **14** 1935–51

[10] Cariñena J F and Ramos A 2005 Lie systems and connections in fibre bundles: applications in quantum mechanics *9th Int. Conf. on Differential Geometry and Its Applications (2004)* ed J Bures (Prague: Matfyzpress) pp 437–52
[11] Cariñena J F, Grabowski J and Ramos A *et al* 2001 Reduction of time-dependent systems admitting a superposition principle *Acta Appl. Math.* **66** 67–87
[12] Cariñena J F, Grabowski J and Marmo G 2007 Superposition rules, Lie theorem and partial differential equations *Rep. Math. Phys.* **60** 237–58
[13] Cariñena J F, de Lucas J and Rañada M F 2008 Recent applications of the theory of Lie systems in Milne–Pinney equations *SIGMA* **4** 031
[14] Cariñena J F and de Lucas J 2008 Integrability of Lie systems and some of its applications in physics *J. Phys. A: Math. Theor.* **41** 304029
[15] Zaitsev V F and Polyanin A D 1994 *Discrete-Group Methods for Integrating Equations of Nonlinear Mechanics* (CA, USA: CRC Press)
[16] Mavromatos N E and Papavassiliou J 1999 Non-linear dynamics in QED_3 and non-trivial infrared structure *Phys. Rev.* D **60** 125008
[17] Momoniat E and Harley C 2006 Approximate implicit solution of a Lane–Emden equation *New Astron.* **11** 520–6
[18] Liu F K 1996 Polytropic gas spheres: an approximate analytic solution of the Lane–Emden equation *Mon. Not. R. Astron. Soc.* **281** 1197
[19] Wong J S W 1975 On the generalized Emden–Fowler equation *SIAM* Review **17** 339–60
[20] Duistermaat J and Kolk J 2000 Lie Groups *Universitext* (New York: Springer)
[21] Wei J and Norman E 1963 Lie algebraic solution of linear differential equations *J. Math. Phys.* **4** 575–81
[22] Panayotounakos D E and Kravvaritis D C 2006 Exact analytic solutions of the Abel, Emden–Fowler and generalized Emden–Fowler nonlinear ODEs *Nonlinear Anal.: Real World Appl.* **7** 634–50
[23] Berkovich L M 1997 The generalized Emden–Fowler equation *Symmetry Nonlinear Math. Phys.* **1** 155–63
[24] Fowler R H 1930 The solution of Emden's and similar differential equations *Mon. Not. R. Astron. Soc.* **91** 63–91
[25] Fowler R H 1931 Further studied of Emden's and similar differential equations *Q. J. Math.* **2** 259–88
[26] Chini M 1898 Sull'equazione del 2• ordine lineare omogenea *Atti. Accad. Sci. Torino* **33** 737–45
[27] Leach P G 1985 First integrals for the modified Emden equation *J. Math. Phys.* **26** 2510–4
[28] Perelomov A 1978 The simple relations between certain dynamical systems *Commun. Math. Phys.* **63** 9–11
[29] Sarlet W and Bahar L Y 1980 A direct construction of first integrals for certain non-linear dynamical systems *Int. J. Nonlinear Mech.* **15** 133
[30] Colegrave R K and Abdalla M S 1983 Invariants for the time-dependent harmonic oscillators *J. Phys. A: Math. Gen.* **12** 3805–15
[31] Mathews P M and Lakshmanan M 1974 On a unique non-linear oscillator *Q. Appl. Math.* **32** 215–8
[32] Lakshmanan M and Rajasekar S 2003 *Nonlinear Dynamics Integrability, Chaos and Patterns, Advanced Texts in Physics* (Berlin: Springer)
[33] Cariñena J F, Rañada M F, Santander M and Senthivelan M 2004 A non-linear oscillator with quasi-harmonic behaviour: two- and n-dimensional oscillator *Nonlinearity* **17** 1941–63
[34] Milson R 1998 Liouville transformation and exactly solvable Schrödinger equation *Int. J. Theor. Phys.* **37** 1735–52
[35] Cariñena J F and de Lucas J 2009 Applications of Lie systems in dissipative Milne–Pinney equations *Int. J. Geom. Methods Mod. Phys.* **6** 683–99
[36] Cariñena J F, Rañada M F and Santander M 2004 A super-integrable two-dimensional non-linear oscillator with an exactly solvable quantum analog *Nonlinearity* **17** 1941–63
[37] Redheffer R 1999 Steen's equation and its generalisations *Aequ. Math.* **58** 60–72